

A possible quantum fluid-dynamical approach to vortex motion in nuclei

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Dedicated to the Memory of Toshio Marumori

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Abstract

The essential point of Bohr-Mottelson theory is to assume a irrotational flow. As was already suggested by Marumori and Watanabe, the internal rotational motion, i.e., the vortex motion, however, may exist also in nuclei. So, we have a necessity of taking the vortex motion into consideration. In a classical fluid dynamics, there are various ways to treat the internal rotational velocity. The Clebsch representation, $\mathbf{v}(\mathbf{x}) = -\nabla\phi(\mathbf{x}) + \lambda(\mathbf{x})\nabla\psi(\mathbf{x})$ (ϕ : velocity potential, λ and ψ : Clebsch parameters) is very powerful and has an advantage deriving equations of fluid motion from a Lagrangian. Making the best use of the advantage, Kronig-Thellung, Ziman and Ito obtained a Hamiltonian including the internal rotational motion, the vortex motion, through the term $\lambda(\mathbf{x})\nabla\psi(\mathbf{x})$. Going to quantum fluid dynamics, Ziman and Thellung finally derived a roton spectrum of liquid Helium II postulated by Landau. Is it possible to apply such the manner to a description of the collective vortex motion in nuclei? The description of such a collective motion has never been treated in the Bohr-Mottelson model for a long time. In this paper, we will investigate a possibility of describing the vortex motion in nuclei basing on the theories of Ziman and Ito together with Marumori's work.

Keywords: Collective motion in nuclei; velocity operator; vortex motion

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1 Introduction

An *exact* treatment of collective variables in nuclei has been attempted [1, 2, 3]. In [1] and [2], Marumori *et al.* first gave a foundation of unified model of collective motion and independent particle motion in nuclei. Applying the Tomonaga's basic idea in his collective motion theory [4] to nuclei, with the aid of the Sunakawa's integral equation method [5], one of the present authors (S.N.) developed collective descriptions of nuclear surface oscillations [6] and of two-dimensional nuclei [7]. These descriptions are considered to give one of possible microscopic foundations of nuclear collective motion derived from the Bohr-Mottelson model [8, 9, 10]. Introducing collective variables, these descriptions were formulated by using the first quantized language, contrary to the second quantized approach in Sunakawa's method. Extending the Tomonaga's idea, Miyazima-Tamura [11, 12] proposed a collective description of nuclear surface oscillations. The other attempt was proposed from another viewpoint, the canonical transformation theory [13]. To approach elementary excitations in a one-dimensional Fermi system, Tomonaga brought a revolutionary idea to the collective motion theory [4]. The Sunakawa's method applicable also to a Fermi system may work well for such a problem to which we have developed an *exact* canonically momenta approach to neutron-proton systems [14].

According to Bohr and Mottelson [15], a nucleus is considered to be a portion of nuclear matter resembling to an incompressible classical fluid but having a sharp surface. Owing to the assumption of a small surface deformation, collective coordinates of the surface oscillations can be described as $R(\theta, \varphi) = R_0 \{1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\theta, \varphi)\}$ (R_0 : Nuclear equilibrium radius). Expanding the collective coordinates around an equilibrium, $\alpha_{\lambda\mu} = 0$, the surface Hamiltonian is given as $H = \sum_{\lambda\mu} \left\{ \frac{1}{2} B_\lambda |\dot{\alpha}_{\lambda\mu}|^2 + \frac{1}{2} C_\lambda |\alpha_{\lambda\mu}|^2 \right\}$ ($\omega_\lambda = \sqrt{\frac{C_\lambda}{B_\lambda}}$) which is a set of harmonic oscillators with frequencies ω_λ . The parameters B_λ and C_λ represent, respectively, the mass parameter associated with collective flow and the nuclear deformability. The essential point of their theory is to assume a **irrotational** flow. Let us introduce a velocity potential $\phi(\mathbf{x}, t)$. The velocity field $\mathbf{v}(\mathbf{x}, t) = -\nabla\phi(\mathbf{x}, t)$ plays a central role over the whole theory. As was already suggested by Marumori *et al.* [3] and Watanabe [13], the **internal rotational** motion, i.e., the vortex motion, however, may exist also in nuclei. So, it is necessary to take the vortex motion into consideration. In a classical fluid dynamics, there are various ways to treat the **internal rotational velocity** $\mathbf{v}(\mathbf{x}, t)$. For this aim, the Clebsch representation [16], $\mathbf{v}(\mathbf{x}, t) = -\nabla\phi(\mathbf{x}, t) + \lambda(\mathbf{x}, t)\nabla\psi(\mathbf{x}, t)$ (λ and ψ : Clebsch parameters) is useful and has an advantage deriving equations of fluid motion from a fluid Lagrangian. Making the maximum use of such the advantage, Kronig-Thellung [17], Ziman [18] and Ito [19] obtained a Hamiltonian including the **internal rotational** motion, the vortex motion, via $\lambda(\mathbf{x}, t)\nabla\psi(\mathbf{x}, t)$. By quantization of fluid dynamics, Ziman and Thellung [20] finally derived a roton spectrum of liquid Helium II postulated by Landau [21]. Is it possible to apply such procedure to a description of the collective vortex motion in nuclei? The description of such a collective motion has never been treated in the Bohr-Mottelson model for a long time. In this paper, we will investigate a possibility of describing the vortex motion in nuclei based on the theories of Ziman and Ito together with Marumori's work.

In §2 we give a brief recapitulation of classical fluid dynamics in terms of Clebsch variables. In §3 we show that a unitary transformation of an A -particle Hamiltonian leads to particle and collective Hamiltonians. The collective Hamiltonian consists of **irrotational** and **internal rotational** motions and their interaction. §4 is devoted to Ziman transformation and to derive the roton Hamiltonian. §5 is also devoted to determination of Clebsch parameters through a one-form gauge potential. Finally in §6 some discussions and further outlook are given.

2 Recapitulation of classical fluid dynamics in terms of Clebsch representation

For a velocity field $\mathbf{v}(\mathbf{x}, t)$ with vortex motion, the Clebsch term $\lambda(\mathbf{x}, t)\nabla\psi(\mathbf{x}, t)$ is useful to derive equation of fluid motion from a Hamilton formalism [22, 23, 16]. Such a representation is very effective in passing through classical fluid dynamics to quantum fluid dynamics. The Clebsch representation is given by

$$\mathbf{v}(\mathbf{x}, t) = -\nabla\phi(\mathbf{x}, t) + \lambda(\mathbf{x}, t)\nabla\psi(\mathbf{x}, t). \quad (2.1)$$

Then the vorticity $\mathbf{w}(\mathbf{x}, t)$ becomes

$$\mathbf{w}(\mathbf{x}, t) = \text{rot}\mathbf{v}(\mathbf{x}, t) = \nabla\lambda(\mathbf{x}, t) \times \nabla\psi(\mathbf{x}, t), \quad (2.2)$$

which generally does not vanish. For simplicity, we denote $\mathbf{v}(\mathbf{x}, t)$ etc. simply as \mathbf{v} etc. In the Clebsch transformation (2.1), it is possible to choose λ and ψ so that the surfaces $\lambda = \text{const.}$ and $\psi = \text{const.}$ [22] move with the fluid, i.e.,

$$\frac{D\lambda}{Dt} = \dot{\lambda} + \mathbf{v} \cdot \nabla\lambda = 0, \quad \frac{D\psi}{Dt} = \dot{\psi} + \mathbf{v} \cdot \nabla\psi = 0, \quad (2.3)$$

where $\frac{D}{Dt}$ is the substantial derivative [24, 23]. Let us start from the Euler equation of fluid,

$$\int \rho \frac{D\mathbf{v}}{Dt} d\tau = \int (\rho \mathbf{F}(\mathbf{x}) - \nabla p) d\tau \rightarrow \frac{D\mathbf{v}}{Dt} = \mathbf{F} - \frac{1}{\rho} \nabla p, \quad \left(\mathbf{F} = -\nabla U \text{ and } \nabla P = \frac{1}{\rho} \nabla p \right). \quad (2.4)$$

where ρ and p are density and pressure of the fluid and \mathbf{F} is external force. The Lagrange differentiation (substantial derivative) for the velocity, $\frac{D\mathbf{v}}{Dt}$, is computed as

$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2} \nabla \mathbf{v}^2 + \dot{\mathbf{v}} - \mathbf{v} \times \text{rot}\mathbf{v} = -\nabla(P + U), \quad \left(P = \int_{p_0}^p \frac{1}{\rho(p)} dp \right), \quad (2.5)$$

Assuming $\mathbf{F} = 0$, with the aid of the vorticity relation (2.2) and the the substantial derivatives for λ and ψ , (2.3), the sum of the second and third terms in the L.H.S. (2.5) is calculated as

$$\begin{aligned} \dot{\mathbf{v}} - \mathbf{v} \times \text{rot}\mathbf{v} &= -\nabla\dot{\phi} + \dot{\lambda}\nabla\psi + \lambda\nabla\dot{\psi} - \mathbf{v} \times (\nabla\lambda \times \nabla\psi) = -\nabla\dot{\phi} + (\dot{\lambda} + \mathbf{v} \cdot \nabla\lambda)\nabla\psi + \nabla(\lambda\dot{\psi}) \\ &= -\nabla(\dot{\phi} - \lambda\dot{\psi}), \end{aligned} \quad (2.6)$$

from which and (2.4), if $U = 0$, we have $\dot{\phi} - \lambda\dot{\psi} = \frac{1}{2}\mathbf{v}^2 + P$. Let us define a Lagrangian density

$$\mathcal{L} = \rho \left(\dot{\phi} - \lambda\dot{\psi} - \frac{1}{2}\mathbf{v}^2 \right) - E_{\text{pot}}(\rho), \quad E_{\text{pot}}(\rho) \equiv \rho \int_{\rho_0}^{\rho} \frac{p - p_0}{\rho^2} d\rho. \quad (2.7)$$

The conjugate momentum to ϕ and ψ are expressed as $\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \rho$ and $\pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = -\rho\lambda$.

Then the Hamiltonian density is given as

$$\mathcal{H} = \pi_\phi \dot{\phi} + \pi_\psi \dot{\psi} - \mathcal{L} = \frac{1}{2} \rho \left(-\nabla\phi - \frac{\pi_\psi}{\rho} \nabla\psi \right)^2 + E_{\text{pot}}(\rho). \quad (2.8)$$

Thus we have the canonical equations of motion $\dot{\rho} = -\frac{\delta \mathcal{H}}{\delta \phi}$, $\dot{\phi} = \frac{\delta \mathcal{H}}{\delta \rho}$, $\dot{\psi} = \frac{\delta \mathcal{H}}{\delta \pi_\psi}$ and $\dot{\pi}_\psi = -\frac{\delta \mathcal{H}}{\delta \psi}$.

From the first equation we can derive the continuity equation of the fluid as follows:

$$\dot{\rho} = \sum_k \frac{\partial}{\partial x_k} \left(\frac{\partial \mathcal{H}}{\partial \partial_k \phi} \right) = \sum_k \frac{\partial}{\partial x_k} (-\rho v_k) = -\text{div}(\rho \mathbf{v}), \quad (x_k = x, y, z, \quad \partial_k \phi = \frac{\partial \phi}{\partial x_k}, \quad k = 1, 2, 3). \quad (2.9)$$

The second, third and forth lead to Eq. (2.3). Further the second also gives $\dot{\phi} - \lambda\dot{\psi} = \frac{1}{2}\mathbf{v}^2 + P$. In the following Sections, the classical scalar fields $\phi(\mathbf{x}, t)$, $\lambda(\mathbf{x}, t)$ and $\psi(\mathbf{x}, t)$ are treated as the corresponding quantal field operators of quantized fluid and then the vector field $\mathbf{v}(\mathbf{x}, t)$ becomes the quantal velocity operator.

3 Unitary transformation of an A -particle Hamiltonian

In an A -particle system with variables $(\mathbf{x}_n, \mathbf{p}_n)$, a Hamiltonian of the system is given by

$$H = \frac{1}{2} \sum_{n=1}^A \frac{\mathbf{p}_n^2}{2m} + V(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_A), \quad (V: \text{Interaction potential}). \quad (3.1)$$

For convenience, from now we omit the time argument. So, we here write ρ etc. as $\rho(\mathbf{x})$ etc. We introduce a unitary operator (UOp) whose exponent has a symmetrized form of operators,

$$U = \exp \left\{ -\frac{i}{\hbar} \int \{ \rho(\mathbf{x}) \phi(\mathbf{x}) - \rho(\mathbf{x}) \lambda(\mathbf{x}) \psi(\mathbf{x}) + \text{Sym.} \} d\mathbf{x} \right\}, \quad \rho(\mathbf{x}) = m \sum_{n=1}^A \delta(\mathbf{x} - \mathbf{x}_n), \quad (3.2)$$

which is an extension of UOp in Ref.[2] to the UOp having the Clebsch variables. The operators $\rho(\mathbf{x})$ and $\phi(\mathbf{x})$ (given through a symmetrized form of (2.1); the quantal velocity operator), canonical conjugate variables, are density and velocity potential of a quantized fluid which obey the canonical equations of motion of the quantized fluid and satisfy the commutation relation $[\rho(\mathbf{x}), \phi(\mathbf{x}')] = i\hbar \delta(\mathbf{x} - \mathbf{x}')$. The operators $-\rho(\mathbf{x})\lambda(\mathbf{x})$ and $\psi(\mathbf{x})$ are also in quite the same situation as the above. As the operator \mathbf{p}_n is given by $\mathbf{p}_n = \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}_n}$, we have the unitary transformations of \mathbf{p}_n and $\frac{\mathbf{p}_n^2}{2m}$ as

$$\left\{ \begin{aligned} U \mathbf{p}_n U^{-1} &= \mathbf{p}_n + \left[U, \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}_n} \right] U^{-1} = \mathbf{p}_n - \frac{\hbar}{i} \frac{\partial U}{\partial \mathbf{x}_n} U^{-1}, \\ U \frac{\mathbf{p}_n^2}{2m} U^{-1} &= \frac{\mathbf{p}_n^2}{2m} + \left[U, \frac{1}{2m} \mathbf{p}_n^2 \right] U^{-1} = \frac{\mathbf{p}_n^2}{2m} + \frac{1}{2m} \frac{\hbar}{i} \left\{ \mathbf{p}_n \frac{\partial U}{\partial \mathbf{x}_n} + \frac{\partial U}{\partial \mathbf{x}_n} \mathbf{p}_n \right\} U^{-1}. \end{aligned} \right. \quad (3.3)$$

Owing to $\frac{\partial}{\partial \mathbf{x}_n} \delta(\mathbf{x} - \mathbf{x}_n) = -\frac{\partial}{\partial \mathbf{x}} \delta(\mathbf{x} - \mathbf{x}_n) = \delta(\mathbf{x} - \mathbf{x}_n) \frac{\partial}{\partial \mathbf{x}}$, we get the gradient formula for U as,

$$\begin{aligned} \frac{\partial U}{\partial \mathbf{x}_n} &= -\frac{i}{\hbar} \left\{ \frac{\partial}{\partial \mathbf{x}_n} \int \left(m \sum_{n'=1}^A \delta(\mathbf{x} - \mathbf{x}_{n'}) \phi(\mathbf{x}) - \lambda(\mathbf{x}) m \sum_{n'=1}^A \delta(\mathbf{x} - \mathbf{x}_{n'}) \psi(\mathbf{x}) + \text{Sym.} \right) d\mathbf{x} \right\} U \\ &= \left\{ -\frac{i}{\hbar} \int \left(m \delta(\mathbf{x} - \mathbf{x}_n) \frac{\partial}{\partial \mathbf{x}} \phi(\mathbf{x}) - \lambda(\mathbf{x}) m \delta(\mathbf{x} - \mathbf{x}_n) \frac{\partial}{\partial \mathbf{x}} \psi(\mathbf{x}) + \text{Sym.} \right) d\mathbf{x} \right\} U. \end{aligned} \quad (3.4)$$

Note that the gradient operator acts onto only canonical conjugate variables ϕ and ψ of ρ and $-\rho\lambda$, respectively. Namely, we here adopt a technically special operator-action rule. Then, the first and second Eqs. of (3.3) read

$$\left. \begin{aligned} U \mathbf{p}_n U^{-1} &= \mathbf{p}_n + \int \{ m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \phi(\mathbf{x}) - \lambda(\mathbf{x}) m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \psi(\mathbf{x}) + \text{Sym.} \} d\mathbf{x}, \\ \sum_{n=1}^A U \frac{\mathbf{p}_n^2}{2m} U^{-1} &= \sum_{n=1}^A \frac{\mathbf{p}_n^2}{2m} + \frac{1}{2} \sum_{n=1}^A \int \{ \{ \mathbf{p}_n \delta(\mathbf{x} - \mathbf{x}_n) + \delta(\mathbf{x} - \mathbf{x}_n) \mathbf{p}_n \} \cdot \nabla \phi(\mathbf{x}) + \text{Sym.} \\ &\quad - \lambda(\mathbf{x}) \{ \mathbf{p}_n \delta(\mathbf{x} - \mathbf{x}_n) + \delta(\mathbf{x} - \mathbf{x}_n) \mathbf{p}_n \} \cdot \nabla \psi(\mathbf{x}) + \text{Sym.} \} d\mathbf{x} + \frac{1}{2} \int \nabla \phi(\mathbf{x}) \cdot \rho(\mathbf{x}) \nabla \phi(\mathbf{x}) d\mathbf{x} \\ &\quad - \frac{1}{2} \int \{ \lambda(\mathbf{x}) \nabla \psi(\mathbf{x}) \cdot \rho(\mathbf{x}) \nabla \phi(\mathbf{x}) + \nabla \phi(\mathbf{x}) \cdot \rho(\mathbf{x}) \lambda(\mathbf{x}) \nabla \psi(\mathbf{x}) - \lambda(\mathbf{x}) \nabla \psi(\mathbf{x}) \cdot \rho(\mathbf{x}) \lambda(\mathbf{x}) \nabla \psi(\mathbf{x}) \} d\mathbf{x}, \end{aligned} \right\} \quad (3.5)$$

the detailed description for derivation of which is given in Appendix A.

Finally, it turns out that the original Hamiltonian is transformed to a new Hamiltonian $\tilde{H} (= U H U^{-1}) = \tilde{H}_0 + \int \tilde{H}_{int.} d\mathbf{x} + \tilde{H}_{field}$ where $\tilde{H}_0 = \sum_{n=1}^A \frac{\mathbf{p}_n^2}{2m} + V(\mathbf{x}_1, \dots, \mathbf{x}_A)$ and $\tilde{H}_{int.}$ is given by $\tilde{H}_{int.} = \frac{1}{2} \sum_{n=1}^A [\{ \mathbf{p}_n \delta(\mathbf{x} - \mathbf{x}_n) + \delta(\mathbf{x} - \mathbf{x}_n) \mathbf{p}_n \} \cdot \nabla \phi(\mathbf{x}) - \lambda(\mathbf{x}) \{ \mathbf{p}_n \delta(\mathbf{x} - \mathbf{x}_n) + \delta(\mathbf{x} - \mathbf{x}_n) \mathbf{p}_n \} \cdot \nabla \psi(\mathbf{x}) + \text{Sym.}]$. (3.6)

The H_{field} is written as $\int (\mathcal{H}_{phon.} + \mathcal{H}_{rot.} + \mathcal{H}_{int.}) d\mathbf{x}$ in which each Hamiltonian \mathcal{H} is expressed as

$$\left. \begin{aligned} \mathcal{H}_{phon.} &= \frac{1}{2} \nabla \phi(\mathbf{x}) \cdot \rho(\mathbf{x}) \nabla \phi(\mathbf{x}), \quad \mathcal{H}_{rot.} = \frac{1}{2} \lambda(\mathbf{x}) \nabla \psi(\mathbf{x}) \cdot \rho(\mathbf{x}) \lambda(\mathbf{x}) \nabla \psi(\mathbf{x}), \\ \mathcal{H}_{int.} &= -\frac{1}{2} \{ \nabla \psi(\mathbf{x}) \cdot \rho(\mathbf{x}) \nabla \phi(\mathbf{x}) + \nabla \phi(\mathbf{x}) \cdot \rho(\mathbf{x}) \lambda(\mathbf{x}) \nabla \psi(\mathbf{x}) \}. \end{aligned} \right\} \quad (3.7)$$

In the Hamiltonian for the quantized fluid, first and second Hamiltonians in (3.7) contribute to the occurrence of the *phonon* and *roton* spectrums, respectively. The last one gives their interaction. They coincident with the classical fluid Hamiltonian (2.8) in the classical limit.

4 Ziman transformation and roton Hamiltonian

To go from classical fluid dynamics to quantum one, Ziman introduced variables ψ_1 and ψ_2 , $\psi = \frac{\psi_1}{\psi_2}$, $\pi_\psi = -\rho\lambda = -\frac{\psi_2^2}{2}$ and field operators $\Psi = \frac{1}{\sqrt{2\hbar}}(\psi_1 + i\psi_2)$ and $\Psi^* = \frac{1}{\sqrt{2\hbar}}(\psi_1 - i\psi_2)$ [18].

Putting the relations into (2.1) and using the operators, the fluid velocity \mathbf{v} is expressed as

$$\mathbf{v} = -\nabla\phi - \frac{1}{2\rho}(\psi_1\nabla\psi_2 - \psi_2\nabla\psi_1) = -\nabla\phi + \frac{i\hbar}{2\rho}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*), \quad (4.1)$$

and also into the second Hamiltonian of (3.7), the roton Hamiltonian $\mathcal{H}_{rot.}$ is obtained as

$$\mathcal{H}_{rot.} = \frac{\hbar^2}{8\rho} \{ (\Psi^*\nabla\Psi \cdot \Psi\nabla\Psi^* + \Psi\nabla\Psi^* \cdot \Psi^*\nabla\Psi) - (\Psi^{*2}\nabla\Psi \cdot \nabla\Psi + \Psi^2\nabla\Psi^* \cdot \nabla\Psi^*) \} \equiv \mathcal{H}_{rot.I} + \mathcal{H}_{rot.II}. \quad (4.2)$$

For the incompressible fluid, due to the continuity equation of the fluid, we have the condition

$$\text{div}\mathbf{v} = -\nabla^2\phi + \frac{i\hbar}{2\rho_0}(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*) = 0, \quad (\rho_0 : \text{equilibrium density}). \quad (4.3)$$

In the Bohr-Mottelson model (BMM), collective flow in nuclei is assumed to be **irrotational**. Namely, the velocity potential satisfies $\nabla^2\phi = 0$ which, using (4.3), leads to $\nabla^2\Psi = \nabla^2\Psi^* = 0$. Then the Ψ and Ψ^* can be expanded in term of spherical harmonic functions as follows:

$$\Psi(\mathbf{x}) = \sum_{\lambda\mu} b_{\lambda\mu} \left(\frac{r}{R_0}\right)^\lambda Y_{\lambda\mu}(\theta, \varphi), \quad \Psi^*(\mathbf{x}) = \sum_{\lambda\mu} b_{\lambda\mu}^* \left(\frac{r}{R_0}\right)^\lambda Y_{\lambda\mu}^*(\theta, \varphi), \quad (4.4)$$

where $b_{\lambda\mu}$ and $b_{\lambda\mu}^*$ are regarded as the boson annihilation and creation operators satisfying $[b_{\lambda\mu}, b_{\lambda'\mu'}^*] = \delta_{\lambda\lambda'}\delta_{\mu\mu'}$, $[b_{\lambda\mu}, b_{\lambda'\mu'}] = [b_{\lambda\mu}^*, b_{\lambda'\mu'}^*] = 0$. The commutation relation between Ψ and Ψ^* are assumed to be canonical, i.e., $[\Psi(\mathbf{x}), \Psi^*(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}')$. The corresponding Poisson brackets in fluid dynamics were discussed in detail by Zakharov-Kuznetsov [25]. The $b_{\lambda\mu}$ and $b_{\lambda\mu}^*$ are roton operators proposed by Landau [21]. Ziman obtained the roton Hamiltonian in terms of them and derived a roton spectrum of liquid Helium. It also acquired the roton Hamiltonian but with a different way from Ziman's. Adopting a vector potential \mathbf{A} obeying the Poisson equation $\nabla^2\mathbf{A} = -\mathbf{w}$ for vorticity \mathbf{w} , \mathbf{A} is represented as $\mathbf{A} = \frac{1}{4\pi} \int \frac{\mathbf{w}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$ [19].

There exists an invariant integral I , $I = \int \mathbf{v} \cdot \mathbf{w} d\mathbf{x} = \frac{1}{4\pi} \iint \frac{(\mathbf{x} - \mathbf{x}') \cdot [\mathbf{w}(\mathbf{x}) \times \mathbf{w}(\mathbf{x}')] }{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x} d\mathbf{x}'$, given in terms of circulations on vortex curves C_i and C_j with strengths κ_i and κ_j . The I is rewritten as $I = \sum_{i,j} \alpha_{i,j} \kappa_i \kappa_j$, $\alpha_{i,j} \equiv \frac{1}{4\pi} \oint_{C_i} \oint_{C_j} \frac{(\mathbf{x} - \mathbf{x}') \cdot [d\mathbf{l}_i \times d\mathbf{l}_j]}{|\mathbf{x} - \mathbf{x}'|^3}$ ($\alpha_{i,j}$: winding number) [26, 27, 28].

First we consider an angular momentum of A -particle system. The total angular momentum of the system is the summation of individual particle angular momentum. Then we have

$$J \equiv \sum_{n=1}^A j_n = \sum_{n=1}^A \mathbf{x}_n \times \mathbf{p}_n = \sum_{n=1}^A \frac{\hbar}{i} \mathbf{x}_n \times \nabla_n \quad (4.5)$$

By a unitary transformation of J , a new total angular momentum $\tilde{J} (= UJU^{-1})$ is changed to

$$\tilde{J} = \sum_{n=1}^A U j_n U^{-1} = \sum_{n=1}^A \{ j_n + [U, j_n] U^{-1} \} = \sum_{n=1}^A \left\{ j_n - \frac{\hbar}{i} \mathbf{x}_n \times (\nabla_n U) U^{-1} \right\}. \quad (4.6)$$

$\nabla_n U$ is given by (3.4). Define \mathbf{L} as $\mathbf{L} \equiv \frac{\hbar}{i} \mathbf{x} \times \nabla$ having a similar form with (4.5). Thus, we get

$$\tilde{J} = J + \int \frac{i}{\hbar} \{ \rho(\mathbf{x}) \mathbf{L} \phi(\mathbf{x}) - \rho(\mathbf{x}) \lambda(\mathbf{x}) \mathbf{L} \psi(\mathbf{x}) \} d\mathbf{x}, \quad (4.7)$$

which consists of the old one J and the angular momentum occurred from the **irrotational** and **rotational** flows. Thus we can find the angular momentum \hat{J} due to the **rotational** flow,

$$\hat{J} = -\frac{i}{\hbar} \rho \lambda \mathbf{L} \psi = \frac{i}{\hbar} \pi_\psi \mathbf{L} \psi = \frac{1}{2} (\Psi \mathbf{L} \Psi^* - \Psi^* \mathbf{L} \Psi), \quad \hat{J}_k \equiv \frac{1}{2} (\Psi L_k \Psi^* - \Psi^* L_k \Psi). \quad (4.8)$$

The spherical tensor representation of $L_k (k = \pm, 0)$ is given as $L_{\pm 1} = \mp \frac{1}{\sqrt{2}} (L_x \pm iL_y)$, $L_0 = L_z$.

Using the Clebsch-Gordan coefficient $\langle l_1 m_1 l_2 m_2 | l_3 m_3 \rangle$, $\Psi L_k \Psi^*$ and $\Psi^* L_k \Psi$ are calculated as

$$\begin{aligned}
\Psi L_k \Psi^* &= \sum_{\lambda' \mu'} b_{\lambda' \mu'} \left(\frac{r}{R_0} \right)^{\lambda'} Y_{\lambda' \mu'} \sum_{\lambda \mu} b_{\lambda \mu}^* \left(\frac{r}{R_0} \right)^{\lambda} L_k Y_{\lambda \mu}^* \\
&= \sum_{\lambda' \mu'} \sum_{\lambda \mu} b_{\lambda' \mu'} b_{\lambda \mu}^* \left(\frac{r}{R_0} \right)^{\lambda + \lambda'} Y_{\lambda' \mu'} (-1)^\mu L_k Y_{\lambda - \mu} \\
&= \sum_{\lambda' \mu'} \sum_{\lambda \mu} b_{\lambda' \mu'} b_{\lambda \mu}^* \left(\frac{r}{R_0} \right)^{\lambda + \lambda'} (-1)^{\mu + k} \sqrt{\lambda(\lambda + 1)} \langle \lambda k - \mu 1 - k | \lambda - \mu \rangle Y_{\lambda' \mu'} Y_{\lambda k - \mu} \quad (4.9) \\
&= \sum_{\lambda' \mu'} \sum_{\lambda \mu} b_{\lambda' \mu'} b_{\lambda \mu}^* (-1)^{\mu + k} \left(\frac{r}{R_0} \right)^{\lambda + \lambda'} \sqrt{\lambda(\lambda + 1)} \langle \lambda k - \mu 1 - k | \lambda - \mu \rangle \\
&\quad \times \left[\sum_{LM} \sqrt{\frac{(2\lambda + 1)(2\lambda' + 1)}{4\pi(2L + 1)}} \langle \lambda k - \mu \lambda' \mu' | LM \rangle \langle \lambda 0 \lambda' 0 | L 0 \rangle Y_{LM} \right]_{M = k - \mu + \mu'}, \\
\Psi^* L_k \Psi &= \sum_{\lambda' \mu'} \sum_{\lambda \mu} b_{\lambda' \mu'}^* b_{\lambda \mu} \left(\frac{r}{R_0} \right)^{\lambda + \lambda'} (-1)^{k + \mu'} \sqrt{\lambda(\lambda + 1)} \langle \lambda \mu + k 1 - k | \lambda \mu \rangle \\
&\quad \times \left[\sum_{LM} \sqrt{\frac{(2\lambda + 1)(2\lambda' + 1)}{4\pi(2L + 1)}} \langle \lambda \mu + k \lambda' - \mu' | LM \rangle \langle \lambda 0 \lambda' 0 | L 0 \rangle Y_{LM} \right]_{M = k - \mu' + \mu} \quad (4.10)
\end{aligned}$$

where the symmetry $\langle \lambda \mu + k 1 - k | \lambda \mu \rangle = (-1)^k \langle \lambda \mu 1 k | \lambda \mu + k \rangle$ and the formula for product of two spherical harmonics are used [29]. Further using the property of the Racah coefficients [29],

$$= \sum_{L' M'} \sqrt{(2\lambda + 1)(2L' + 1)} W(\lambda 1 L \lambda'; \lambda L') \langle 1 k \lambda' - \mu' | L' M' \rangle \langle \lambda \mu L' M' | LM \rangle,$$

and substituting (4.9) and (4.10) into (4.8), the k -th component of **rotational** angular momentum \hat{J}_k is derived as

$$\begin{aligned}
\hat{J}_k &= \frac{1}{2} \sum_{\lambda \mu \lambda' \mu'} \sum_{LM L' M'} (-1)^{\lambda + \lambda' + 1} (2\lambda + 1) \sqrt{\frac{\lambda(\lambda + 1)(2\lambda' + 1)(2L' + 1)}{12\pi}} \left(\frac{r}{R_0} \right)^{\lambda + \lambda'} \\
&\quad \times W(\lambda 1 L L'; \lambda \lambda') \langle \lambda 0 \lambda' 0 | L 0 \rangle Y_{LM} \langle LM \lambda \mu | L' M' \rangle \langle L' M' \lambda' \mu' | 1 k \rangle \\
&\quad \times [b_{\lambda \mu}^* (-1)^{\mu'} b_{\lambda' - \mu'} - b_{\lambda' \mu'}^* (-1)^\mu b_{\lambda - \mu} + \delta_{\lambda \lambda'} (-1)^\mu \delta_{\mu - \mu'}], \quad (4.11)
\end{aligned}$$

whose form, neglecting the constant term, is very similar to the angular momentum given in terms of quadratic surface-phonon operators in the BMM. Then, the above **rotational** angular momentum \hat{J} may have, in some sense, an intimate connection with the "vortex angular momentum" proposed by Cusson, which is proportional to the vorticity [30].

Finally, using the gradient formula [31, 29],

$$\nabla_k \left(\frac{r}{R_0} \right)^{\lambda} Y_{\lambda \mu} = -\sqrt{\frac{\lambda}{2\lambda - 1}} \langle \lambda \mu 1 k | \lambda - 1 \mu + k \rangle \frac{2\lambda + 1}{R_0} \left(\frac{r}{R_0} \right)^{\lambda - 1} Y_{\lambda - 1 \mu + k},$$

and noticing the fact that the scalar product of any two first-rank tensors \mathbf{v} and \mathbf{v}' is given in the spherical tensor representation as $\sum_k (-1)^k v_{1k} v'_{1-k}$, we obtain the roton Hamiltonian $\mathcal{H}_{rot.} (\equiv \mathcal{H}_{rot.I} + \mathcal{H}_{rot.II})$ (4.2) in terms of the roton operators $b_{\lambda \mu}$ and $b_{\lambda \mu}^*$ as,

$$\begin{aligned}
\mathcal{H}_{rot.I} &= \frac{\hbar^2}{8\rho_0} \sum_k \sum_{\lambda \lambda' \kappa \kappa' \mu \mu' \nu \nu'} \sum_{LL' MM' \Gamma \Gamma' \Lambda \Lambda'} \sum_{JK} (-1)^{k+J} \frac{1}{R_0^2} \left(\frac{r}{R_0} \right)^{\lambda + \lambda' + \kappa + \kappa' - 2} \frac{(2\lambda' + 1)(2\kappa' + 1)}{3} \\
&\quad \times \sqrt{\frac{(2\lambda + 1)\lambda'(2\lambda' - 1)(2\Gamma + 1)}{4\pi}} \sqrt{\frac{(2\kappa + 1)\kappa'(2\kappa' - 1)(2\Gamma' + 1)}{4\pi}} \sqrt{\frac{(2L + 1)(2L' + 1)}{4\pi(2J + 1)}} \\
&\quad \times \langle \lambda 0 \lambda' - 1 0 | L 0 \rangle \langle \kappa 0 \kappa' - 1 0 | L' 0 \rangle \langle L 0 L' 0 | J 0 \rangle \langle LM \lambda' \mu' | \Gamma \Lambda \rangle \langle L' M' \kappa' \nu' | \Gamma' \Lambda' \rangle \langle LML'M' | JK \rangle Y_{JK} \\
&\quad \times \langle \Gamma \Lambda \lambda \mu | 1 k \rangle \langle \Gamma' \Lambda' \kappa \nu | 1 - k \rangle W(\lambda' \lambda' - 1 \Gamma \lambda; 1 L) W(\kappa' \kappa' - 1 \Gamma' \kappa; 1 L') \\
&\quad \times [b_{\lambda \mu}^* b_{\lambda' \nu'}^* (-1)^{\mu'} b_{\lambda - \mu'} (-1)^{\nu} b_{\kappa - \nu} + \delta_{\kappa \kappa'} (-1)^{\nu} \delta_{-\nu, \nu'} b_{\lambda \mu}^* (-1)^{\mu'} b_{\lambda - \mu'} + \delta_{\lambda' \kappa'} (-1)^{\mu'} \delta_{-\mu', \nu'} b_{\lambda \mu}^* (-1)^{\nu} b_{\kappa - \nu} \\
&\quad + b_{\lambda' \mu'}^* b_{\kappa \nu}^* (-1)^{\mu} b_{\lambda - \mu} (-1)^{\nu'} b_{\kappa' - \nu'} + \delta_{\lambda \lambda'} (-1)^{\mu} \delta_{-\mu, \mu'} b_{\lambda \mu}^* (-1)^{\nu'} b_{\kappa' - \nu'} + \delta_{\lambda \kappa} (-1)^{\mu} \delta_{-\mu, \nu} b_{\lambda' \mu'}^* (-1)^{\nu'} b_{\kappa' - \nu'}], \quad (4.12)
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_{rot.II} = & \frac{\hbar^2}{8\rho_0} \sum_k \sum_{\lambda\lambda'\kappa\kappa'\mu\mu'\nu\nu'} \sum_{LL'MM'\Gamma\Gamma'\Lambda\Lambda'} \sum_{JK} \frac{1}{R_0^2} \left(\frac{r}{R_0}\right)^{\lambda+\lambda'+\kappa+\kappa'-2} (-1)^{\kappa+\kappa'} \frac{(2\lambda'+1)(2\kappa'+1)}{\sqrt{3}} \\
& \times \sqrt{\frac{(2\lambda+1)(2\lambda'+1)}{4\pi(2L+1)}} \sqrt{\frac{\kappa\kappa'(2\kappa'-1)(2\Gamma'+1)}{4\pi}} \sqrt{\frac{(2L+1)(2L'+1)}{4\pi(2J+1)}} \\
& \times \langle \lambda 0 \lambda' 0 | L 0 \rangle \langle \kappa - 10 \kappa' - 10 | L' 0 \rangle \langle L 0 L' 0 | J 0 \rangle \langle \lambda \mu \lambda' \mu' | L M \rangle (-1)^{\nu'} \langle L' M' \kappa' \nu' | \Gamma' \Lambda' \rangle \langle L M L' M' | J K \rangle Y_{JK} \\
& \times \langle \Gamma' \Lambda' \kappa - 1 \nu + k | 1 k \rangle W(\kappa' \kappa' - 1 \Gamma' \kappa - 1; 1 L') \\
& \times [(-1)^\mu b_{\lambda-\mu}^* (-1)^{\mu'} b_{\lambda'-\mu'}^* (-1)^\nu b_{\kappa-\nu} (-1)^{\nu'} b_{\kappa'-\nu'} + b_{\kappa\nu}^* b_{\kappa'\nu'}^* b_{\lambda\mu} b_{\lambda'\mu'} + \delta_{\lambda\kappa} \delta_{\mu\mu'} b_{\kappa'\nu'}^* b_{\lambda'\nu} + \delta_{\lambda\kappa} \delta_{\mu\nu} b_{\kappa\nu}^* b_{\lambda'\mu'} \\
& + \delta_{\lambda'\kappa} \delta_{\mu'\nu} b_{\kappa'\nu'}^* b_{\lambda\mu} + \delta_{\lambda'\kappa'} \delta_{\mu'\nu'} b_{\kappa\nu}^* b_{\lambda\mu} + \delta_{\lambda\kappa} \delta_{\mu\nu} \delta_{\lambda'\kappa'} \delta_{\mu'\nu'} + \delta_{\lambda'\kappa} \delta_{\mu'\nu} \delta_{\lambda\kappa'} \delta_{\mu\nu}].
\end{aligned} \tag{4.13}$$

The roton Hamiltonian $\mathcal{H}_{rot.}$ consists of normal-ordered quartic and quadratic terms with respect to roton operators and constant terms. On the contrary the BMM Hamiltonian has a quadratic form of surface-phonon operators. This is a remarkable difference between them.

There exist many kinds of multiple degrees of freedom in the collective coordinates of the surface oscillations in the BMM. As was done in it, we also pay our special interest in collective excitations with quadrupole degrees of freedom since such a degrees of freedom makes a fundamental role in almost all nuclei. Then, in Eqs. (4.11), (4.12) and (4.13), we restrict to only the case of $\lambda = \lambda' = \kappa = \kappa'$ and they are rewritten in the following forms:

The **rotational** angular momentum \hat{J}_k (4.11) at the nuclear surface ($r=R_0$) is expressed as

$$\begin{aligned}
\hat{J}_k = & \frac{5}{2} \sqrt{\frac{2 \cdot 3 \cdot 5 (2L+1)}{12\pi}} \sum_{\mu\mu'} \sum_{LML'M'} W(21LL'; 22) \langle 2020 | L 0 \rangle Y_{LM} \\
& \times \langle LM 2\mu | L' M' \rangle \langle L' M' 2\mu' | 1 k \rangle [b_{2\mu'}^* (-1)^\mu b_{2-\mu} - b_{2\mu}^* (-1)^{\mu'} b_{2-\mu'} - (-1)^{\mu'} \delta_{\mu-\mu'}],
\end{aligned} \tag{4.14}$$

in which we pick up only the term with $L=0$. Then we have a simple formula for \hat{J}_k as

$$\hat{J}_k = -\frac{\sqrt{2 \cdot 5}}{4\pi} \sum_{\mu\mu'} \langle 2\mu 2\mu' | 1 k \rangle b_{2\mu}^* (-1)^{\mu'} b_{2-\mu'}, \tag{4.15}$$

which has the same form as the BMM angular momentum operator except the minus sign but is expressed in terms of the roton operators without constant term.

The roton Hamiltonian $\mathcal{H}_{rot.} (\equiv \mathcal{H}_{rot.I} + \mathcal{H}_{rot.II})$ (4.2) is reduced to more simple forms in terms of the roton operators $b_{2\mu}^*$ and $b_{2\mu}$ as,

$$\begin{aligned}
\mathcal{H}_{rot.I} = & \frac{\hbar^2}{8\rho_0} \frac{1}{R_0^2} \left(\frac{r}{R_0}\right)^6 \frac{5 \cdot 5}{3} \frac{2 \cdot 3 \cdot 5}{4\pi} \sum_k \sum_{\mu\mu'\nu\nu'} \sum_{LL'MM'\Gamma\Gamma'\Lambda\Lambda'} \sum_{JK} (-1)^{k+J} \\
& \times \sqrt{\frac{(2L+1)(2L'+1)(2\Gamma+1)(2\Gamma'+1)}{4\pi(2J+1)}} \langle 2010 | L 0 \rangle \langle 2010 | L' 0 \rangle \langle L 0 L' 0 | J 0 \rangle \langle LM 2\mu' | \Gamma \Lambda \rangle \langle L' M' 2\nu' | \Gamma' \Lambda' \rangle \\
& \times \langle L M L' M' | J K \rangle Y_{JK} \langle \Gamma \Lambda 2\mu | 1 k \rangle \langle \Gamma' \Lambda' 2\nu | 1 - k \rangle W(21\Gamma 2; 1 L) W(21\Gamma' 2; 1 L') \\
& \times [b_{2\mu}^* b_{2\nu'}^* (-1)^{\mu'} b_{2-\mu'} (-1)^\nu b_{2-\nu} + (-1)^\nu \delta_{-\nu,\nu'} b_{2\mu}^* (-1)^{\mu'} b_{2-\mu'} + (-1)^{\mu'} \delta_{-\mu',\nu'} b_{2\mu}^* (-1)^\nu b_{2-\nu} \\
& + b_{2\mu}^* b_{2\nu}^* (-1)^\mu b_{2-\mu} (-1)^{\nu'} b_{2-\nu'} + (-1)^\mu \delta_{-\mu,\mu'} b_{2\nu}^* (-1)^{\nu'} b_{2-\nu'} + (-1)^\mu \delta_{-\mu,\nu} b_{2\mu}^* (-1)^{\nu'} b_{2-\nu'}],
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
\mathcal{H}_{rot.II} = & \frac{\hbar^2}{8\rho_0} \frac{1}{R_0^2} \left(\frac{r}{R_0}\right)^6 \frac{5 \cdot 5}{\sqrt{3}} \frac{2 \cdot \sqrt{3} \cdot 5}{4\pi} \sum_k \sum_{\mu\mu'\nu\nu'} \sum_{LL'MM'\Gamma\Gamma'\Lambda\Lambda'} \sum_{JK} \\
& \times \sqrt{\frac{2\Gamma'+1}{2L+1}} \sqrt{\frac{(2L+1)(2L'+1)}{4\pi(2J+1)}} \langle 2020 | L 0 \rangle \langle 1010 | L' 0 \rangle \langle L 0 L' 0 | J 0 \rangle \langle 2\mu 2\mu' | L M \rangle (-1)^{\nu'} \langle L' M' 2\nu' | \Gamma' \Lambda' \rangle \\
& \times \langle L M L' M' | J K \rangle Y_{JK} \langle \Gamma' \Lambda' 1 \nu + k | 1 k \rangle W(21\Gamma' 1; 1 L') \\
& \times [(-1)^\mu b_{2-\mu}^* (-1)^{\mu'} b_{2-\mu'}^* (-1)^\nu b_{2-\nu} (-1)^{\nu'} b_{2-\nu'} + b_{2\nu}^* b_{2\nu'}^* b_{2\mu} b_{2\mu'} + \delta_{\mu\nu} b_{2\nu'}^* b_{2\nu} + \delta_{\mu\nu'} b_{2\nu}^* b_{2\mu'} \\
& + \delta_{\mu'\nu} b_{2\nu'}^* b_{2\mu} + \delta_{\mu'\nu'} b_{2\nu}^* b_{2\mu} + \delta_{\mu\nu} \delta_{\mu'\nu'} + \delta_{\mu'\nu} \delta_{\mu\nu}].
\end{aligned} \tag{4.17}$$

In (4.16), we also pick up only the term with $L=L'=1$, $J=0$ and $\Gamma=\Gamma'=1$. Then we have

$$\begin{aligned}
& \mathcal{H}_{rot.I} = \frac{\hbar^2}{8\rho_0} \frac{1}{R_0^2} \left(\frac{r}{R_0} \right)^6 \frac{2 \cdot 5}{3(4\pi)^2} \sum_k \sum_{\mu\mu'\nu\nu'} \sum_M (-1)^k (-1)^M \\
& \times \sum_{IK} \sqrt{3(2I+1)} W(1212; 1I) \langle 2\mu' 2\mu | IK \rangle \langle 1MIK | 1k \rangle \\
& \times \sum_{I'K'} \sqrt{3(2I'+1)} W(1212; 1I') \langle 2\nu' 2\nu | I'K' \rangle \langle 1-MI'K' | 1-k \rangle \\
& \times [b_{2\mu}^* b_{2\nu'}^* (-1)^{\mu'} b_{2-\mu'} (-1)^\nu b_{2-\nu} + (-1)^\nu \delta_{-\nu,\nu'} b_{2\mu}^* (-1)^{\mu'} b_{2-\mu'} + (-1)^{\mu'} \delta_{-\mu',\nu'} b_{2\mu}^* (-1)^\nu b_{2-\nu} \\
& + b_{2\mu'}^* b_{2\nu}^* (-1)^\mu b_{2-\mu} (-1)^{\nu'} b_{2-\nu'} + (-1)^\mu \delta_{-\mu,\mu'} b_{2\nu}^* (-1)^{\nu'} b_{2-\nu'} + (-1)^\mu \delta_{-\mu,\nu} b_{2\mu'}^* (-1)^{\nu'} b_{2-\nu'}] \\
& = \frac{\hbar^2}{8\rho_0} \frac{1}{R_0^2} \left(\frac{r}{R_0} \right)^6 \frac{2 \cdot 5}{3(4\pi)^2} \sum_k \sum_{\mu\mu'\nu\nu'} \sum_M (-1)^k (-1)^M \\
& \times \left\{ \frac{1}{5} (-1)^{\mu} \delta_{\mu',-\mu} \delta_{Mk} - \frac{3}{2\sqrt{5}} \sum_K \langle 2\mu' 2\mu | 1K \rangle \langle 1MIK | 1k \rangle \right\} \left\{ \frac{1}{5} (-1)^\nu \delta_{\nu',-\nu} \delta_{Mk} - \frac{3}{2\sqrt{5}} \sum_K \langle 2\nu' 2\nu | 1K' \rangle \langle 1-MI'K' | 1-k \rangle \right\} \\
& \times [b_{2\mu}^* b_{2\nu'}^* (-1)^{\mu'} b_{2-\mu'} (-1)^\nu b_{2-\nu} + (-1)^\nu \delta_{-\nu,\nu'} b_{2\mu}^* (-1)^{\mu'} b_{2-\mu'} + (-1)^{\mu'} \delta_{-\mu',\nu'} b_{2\mu}^* (-1)^\nu b_{2-\nu} \\
& + b_{2\mu'}^* b_{2\nu}^* (-1)^\mu b_{2-\mu} (-1)^{\nu'} b_{2-\nu'} + (-1)^\mu \delta_{-\mu,\mu'} b_{2\nu}^* (-1)^{\nu'} b_{2-\nu'} + (-1)^\mu \delta_{-\mu,\nu} b_{2\mu'}^* (-1)^{\nu'} b_{2-\nu'}] \\
& = \frac{\hbar^2}{8\rho_0} \frac{1}{R_0^2} \left(\frac{r}{R_0} \right)^6 \frac{2}{3 \cdot 5(4\pi)^2} \sum_k \sum_{\mu\mu'\nu\nu'} (-1)^{\mu'} \delta_{-\mu',\mu} (-1)^{\nu'} \delta_{-\nu',\nu} \\
& \times [b_{2\mu}^* b_{2\nu'}^* (-1)^{\mu'} b_{2-\mu'} (-1)^\nu b_{2-\nu} + (-1)^\nu \delta_{-\nu,\nu'} b_{2\mu}^* (-1)^{\mu'} b_{2-\mu'} + (-1)^{\mu'} \delta_{-\mu',\nu'} b_{2\mu}^* (-1)^\nu b_{2-\nu} \\
& + b_{2\mu'}^* b_{2\nu}^* (-1)^\mu b_{2-\mu} (-1)^{\nu'} b_{2-\nu'} + (-1)^\mu \delta_{-\mu,\mu'} b_{2\nu}^* (-1)^{\nu'} b_{2-\nu'} + (-1)^\mu \delta_{-\mu,\nu} b_{2\mu'}^* (-1)^{\nu'} b_{2-\nu'}] \\
& - \frac{\hbar^2}{8\rho_0} \frac{1}{R_0^2} \left(\frac{r}{R_0} \right)^6 \frac{2}{5(4\pi)^2} \sum_{\mu\mu'\nu\nu'} (-1)^\mu \delta_{\mu',-\mu} (-1)^\nu \delta_{\nu',-\nu} \sum_k \langle 1k10 | 1k \rangle \\
& \times [b_{2\mu}^* b_{2\nu'}^* (-1)^{\mu'} b_{2-\mu'} (-1)^\nu b_{2-\nu} + (-1)^\nu \delta_{-\nu,\nu'} b_{2\mu}^* (-1)^{\mu'} b_{2-\mu'} + (-1)^{\mu'} \delta_{-\mu',\nu'} b_{2\mu}^* (-1)^\nu b_{2-\nu} \\
& + b_{2\mu'}^* b_{2\nu}^* (-1)^\mu b_{2-\mu} (-1)^{\nu'} b_{2-\nu'} + (-1)^\mu \delta_{-\mu,\mu'} b_{2\nu}^* (-1)^{\nu'} b_{2-\nu'} + (-1)^\mu \delta_{-\mu,\nu} b_{2\mu'}^* (-1)^{\nu'} b_{2-\nu'}] \\
& + \frac{\hbar^2}{8\rho_0} \frac{1}{R_0^2} \left(\frac{r}{R_0} \right)^6 \frac{3 \cdot 3 \cdot 3}{2 \cdot 4 \cdot 5(4\pi)^2} \sum_k \sum_{\mu\mu'\nu\nu'} \sum_{M\Lambda\Lambda'} (-1)^{k+M} \\
& \times \langle 2\mu 1M | 1\Lambda \rangle \langle 2\mu' 1\Lambda | 1k \rangle \langle 2\nu 1-M | 1\Lambda' \rangle \langle 2\nu' 1\Lambda' | 1-k \rangle \\
& \times [b_{2\mu}^* b_{2\nu'}^* (-1)^{\mu'} b_{2-\mu'} (-1)^\nu b_{2-\nu} + (-1)^\nu \delta_{-\nu,\nu'} b_{2\mu}^* (-1)^{\mu'} b_{2-\mu'} + (-1)^{\mu'} \delta_{-\mu',\nu'} b_{2\mu}^* (-1)^\nu b_{2-\nu} \\
& + b_{2\mu'}^* b_{2\nu}^* (-1)^\mu b_{2-\mu} (-1)^{\nu'} b_{2-\nu'} + (-1)^\mu \delta_{-\mu,\mu'} b_{2\nu}^* (-1)^{\nu'} b_{2-\nu'} + (-1)^\mu \delta_{-\mu,\nu} b_{2\mu'}^* (-1)^{\nu'} b_{2-\nu'}] \\
& = \frac{8}{5} f(r) \sum_\mu b_{2\mu}^* b_{2\mu} + \frac{4}{5} f(r) \sum_\mu b_{2\mu}^* b_{2\mu} \sum_\nu b_{2\nu}^* b_{2\nu} \\
& + \frac{3 \cdot 3 \cdot 3}{2 \cdot 4 \cdot 5} f(r) \sum_k \sum_{\mu\mu'\nu\nu'} \sum_{M\Lambda\Lambda'} (-1)^{k+M} \langle 2\mu 1M | 1\Lambda \rangle \langle 2\mu' 1\Lambda | 1k \rangle \langle 2\nu 1-M | 1\Lambda' \rangle \langle 2\nu' 1\Lambda' | 1-k \rangle \\
& \times [2(-1)^\nu \delta_{-\nu,\nu'} b_{2\mu}^* (-1)^{\mu'} b_{2-\mu'} + (1+(-1)^{\Lambda+\Lambda'}) (-1)^\mu \delta_{-\mu,\nu} b_{2\mu'}^* (-1)^{\nu'} b_{2-\nu'}] \\
& + \frac{3 \cdot 3 \cdot 3}{2 \cdot 4 \cdot 5} f(r) \sum_k \sum_{\mu\mu'\nu\nu'} \sum_{M\Lambda\Lambda'} (-1)^{k+M} \langle 2\mu 1M | 1\Lambda \rangle \langle 2\mu' 1\Lambda | 1k \rangle \langle 2\nu 1-M | 1\Lambda' \rangle \langle 2\nu' 1\Lambda' | 1-k \rangle \\
& \times [b_{2\mu}^* b_{2\nu'}^* (-1)^{\mu'} b_{2-\mu'} (-1)^\nu b_{2-\nu} + b_{2\mu'}^* b_{2\nu}^* (-1)^\mu b_{2-\mu} (-1)^{\nu'} b_{2-\nu'}],
\end{aligned} \tag{4.18}$$

where we have introduced $f(r)$ defined as $f(r) \equiv \frac{1}{(4\pi)^2} \frac{\hbar^2}{8\rho_0} \frac{1}{R_0^2} \left(\frac{r}{R_0} \right)^6$ and used the relation $\sum_k \langle 1k10 | 1k \rangle = 0$ and taken only the terms $I=I'=0$ and $I=I'=1$. We here omit the terms with $I, I' \neq 0, 1$ since they are very lengthy equations.

In (4.17), we also pick up only the term with $L = L' = J = 0$. Then we have

$$\begin{aligned}
\mathcal{H}_{rot.II} = & -\frac{2 \cdot 5 \cdot 5}{3} f(r) \sum_k \sum_{\mu\mu'\nu\nu'} (-1)^\mu \delta_{\mu'-\mu} (-1)^\nu \delta_{\nu'-\nu} \langle 2-\nu 1 \nu+k | 1k \rangle \\
& \times [(-1)^\mu b_{2-\mu}^* (-1)^{\mu'} b_{2-\mu'}^* (-1)^\nu b_{2-\nu} (-1)^{\nu'} b_{2-\nu'} + b_{2\nu}^* b_{2\nu'}^* b_{2\mu} b_{2\mu'} + \delta_{\mu\nu} b_{2\nu}^* b_{2\nu'} + \delta_{\mu\nu'} b_{2\nu}^* b_{2\mu'} \\
& + \delta_{\mu'\nu} b_{2\nu'}^* b_{2\mu} + \delta_{\mu'\nu'} b_{2\nu}^* b_{2\mu} + \delta_{\mu\nu} \delta_{\mu'\nu'} + \delta_{\mu'\nu} \delta_{\mu\nu'}] \\
& = -\frac{4 \cdot 5 \cdot 5}{3} f(r) \sum_k \sum_\nu \langle 2-\nu 1 \nu+k | 1k \rangle \\
& -\frac{2 \cdot 5 \cdot 5}{3} f(r) \sum_k \sum_\nu \{ \langle 2\nu 1k | 1\nu+k \rangle (-1)^\nu b_{2-\nu}^* b_{2\nu} + 3 \langle 2-\nu 1 \nu+k | 1k \rangle b_{2\nu}^* b_{2\nu} \} \\
& -\frac{2 \cdot 5 \cdot 5}{3} f(r) \sum_k \sum_{\mu\nu} \langle 2-\nu 1 \nu+k | 1k \rangle [b_{2\mu}^* (-1)^\mu b_{2-\mu}^* b_{2\nu} (-1)^\nu b_{2-\nu} + b_{2\nu}^* (-1)^\nu b_{2-\nu}^* b_{2\mu} (-1)^\mu b_{2-\mu}].
\end{aligned} \tag{4.19}$$

Adding (4.19) to (4.18), finally we reach to the final expression for the roton Hamiltonian $\mathcal{H}_{rot.}$ given in the following form:

$$\begin{aligned}
\mathcal{H}_{rot.} = & -\frac{100}{3} f(r) \sum_k \sum_\nu \langle 2-\nu 1 \nu+k | 1k \rangle + \frac{8}{5} f(r) \sum_\mu b_{2\mu}^* b_{2\mu} + \frac{4}{5} f(r) \sum_\mu b_{2\mu}^* b_{2\mu} \sum_\nu b_{2\nu}^* b_{2\nu} \\
& -\frac{50}{3} f(r) \sum_k \sum_\nu \{ \langle 2\nu 1k | 1\nu+k \rangle (-1)^\nu b_{2-\nu}^* b_{2\nu} + 3 \langle 2-\nu 1 \nu+k | 1k \rangle b_{2\nu}^* b_{2\nu} \} \\
& + \frac{27}{40} f(r) \sum_k \sum_{\mu\mu'\nu\nu'} \sum_{M\Lambda\Lambda'} (-1)^{k+M} \langle 2\mu 1M | 1\Lambda \rangle \langle 2\mu' 1\Lambda | 1k \rangle \langle 2\nu 1-M | 1\Lambda' \rangle \langle 2\nu' 1\Lambda' | 1-k \rangle \\
& \times [2(-1)^\nu \delta_{-\nu,\nu'} b_{2\mu}^* (-1)^{\mu'} b_{2-\mu'} + (1 + (-1)^{\Lambda+\Lambda'}) (-1)^\mu \delta_{-\mu,\mu'} b_{2\mu'}^* (-1)^{\nu'} b_{2-\nu'}] \\
& -\frac{50}{3} f(r) \sum_k \sum_{\mu\nu} \langle 2-\nu 1 \nu+k | 1k \rangle [b_{2\mu}^* (-1)^\mu b_{2-\mu}^* b_{2\nu} (-1)^\nu b_{2-\nu} + b_{2\nu}^* (-1)^\nu b_{2-\nu}^* b_{2\mu} (-1)^\mu b_{2-\mu}] \\
& + \frac{27}{40} f(r) \sum_k \sum_{\mu\mu'\nu\nu'} \sum_{M\Lambda\Lambda'} (-1)^{k+M} \langle 2\mu 1M | 1\Lambda \rangle \langle 2\mu' 1\Lambda | 1k \rangle \langle 2\nu 1-M | 1\Lambda' \rangle \langle 2\nu' 1\Lambda' | 1-k \rangle \\
& \times [b_{2\mu}^* b_{2\nu'}^* (-1)^{\mu'} b_{2-\mu'} (-1)^\nu b_{2-\nu} + b_{2\mu'}^* b_{2\nu}^* (-1)^\mu b_{2-\mu} (-1)^{\nu'} b_{2-\nu'}] + \dots,
\end{aligned} \tag{4.20}$$

in which we omit terms with higher rank of angular momenta L, L' and J etc. on the way of computing (4.16) and (4.17) because their explicit expressions become too long to write. The roton Hamiltonian at the equilibrium nuclear surface is given by $\mathcal{H}_{rot.}|_{r=R_0}$.

The velocity potential $\phi(\mathbf{x})$ is also expanded as $\phi(\mathbf{x}) = \sum_\mu \pi_{2\mu} (r/R_0)^2 Y_{2\mu}(\theta, \varphi)$ where $\pi_{2\mu}$ and the collective coordinate $\alpha_{2\mu}$ given previously are related to the BMM boson operator. To get the fluid velocity $\mathbf{v}(\mathbf{x})$ at the surface, we also require a surface boundary condition $\partial R(\theta, \varphi)/\partial t|_{r=R_0} = v_r|_{r=R_0}$. Using the fluid velocity \mathbf{v} (4.1), the condition is given as follows:

$$\begin{aligned}
R_0 \sum_\mu \dot{\alpha}_{2\mu} Y_{2\mu} = & - \sum_\mu \pi_{2\mu} \frac{\partial}{\partial r} \left(\frac{r}{R_0} \right)^2 Y_{2\mu} \Big|_{r=R_0} \\
& - \frac{i\hbar}{2\rho_0} \sum_{\mu\mu'} \left\{ b_{2\mu}^* \left(\frac{r}{R_0} \right)^2 Y_{2\mu}^* b_{2\mu'} \frac{\partial}{\partial r} \left(\frac{r}{R_0} \right)^2 Y_{2\mu'} \Big|_{r=R_0} - b_{2\mu} \left(\frac{r}{R_0} \right)^2 Y_{2\mu} b_{2\mu'}^* \frac{\partial}{\partial r} \left(\frac{r}{R_0} \right)^2 Y_{2\mu'}^* \Big|_{r=R_0} \right\} \\
& = -\frac{2}{R_0} \sum_\mu \pi_{2\mu} Y_{2\mu} - \frac{i\hbar}{2\rho_0} \frac{2}{R_0} \sum_\mu \left\{ \left(\sum_{\mu'} (-1)^{\mu'} b_{2-\mu'}^* Y_{2\mu'} \right) b_{2\mu} - \left(\sum_{\mu'} b_{2\mu'} Y_{2\mu'} \right) (-1)^\mu b_{2-\mu}^* \right\} Y_{2\mu},
\end{aligned} \tag{4.21}$$

which gives the relation between the time derivative $\dot{\alpha}_{2\mu}$ and the $\pi_{2\mu}$ and roton operator $b_{2\mu}$ as

$$\dot{\alpha}_{2\mu} = -\frac{2}{R_0^2} \pi_{2\mu} - \frac{i\hbar}{2\rho_0} \frac{2}{R_0^2} \left\{ \left(\sum_{\mu'} (-1)^{\mu'} b_{2-\mu'}^* Y_{2\mu'} \right) b_{2\mu} - \left(\sum_{\mu'} b_{2\mu'} Y_{2\mu'} \right) (-1)^\mu b_{2-\mu}^* \right\}. \tag{4.22}$$

It should be noticed that in the above there exist bi-linear terms of $b_{2\mu}$ and $(-1)^\mu b_{2-\mu}^*$. This is a remarkable point of the relation. However, if we discard these terms, we can reach to the well-known result that the $\pi_{2\mu}$ is regarded as a canonical conjugate variable to the $\alpha_{2\mu}$.

5 Determination of Clebsch parameters through a one-form gauge potential

Using the equilibrium density ρ_0 , the Ziman variables ψ_1 and ψ_2 are expressed through the two Clebsch parameters ψ and λ as

$$\left. \begin{aligned} \psi &= \frac{\psi_1}{\psi_2}, \quad \psi_1 = \psi\psi_2 = \psi\sqrt{2\rho_0\lambda}, \\ \rho_0\lambda &= \frac{\psi_2^2}{2}, \quad \psi_2 = \sqrt{2\rho_0\lambda}. \end{aligned} \right\} \quad (5.1)$$

Substituting (5.1) into the Ziman field operators $\Psi = \frac{1}{\sqrt{2\hbar}}(\psi_1 + i\psi_2)$ and $\Psi^* = \frac{1}{\sqrt{2\hbar}}(\psi_1 - i\psi_2)$, then we have the same expression as the Allcock-Kuper's one [32] for Ψ and Ψ^* as follows:

$$\left. \begin{aligned} \Psi &= \sqrt{\frac{\rho_0\lambda}{\hbar}}(\psi + i), \\ \Psi^* &= \sqrt{\frac{\rho_0\lambda}{\hbar}}(\psi - i). \end{aligned} \right\} \quad (5.2)$$

For the incompressible fluid, the continuity equation of the fluid leads us to the condition

$$\text{div} \mathbf{v} = -\nabla^2 \phi + \frac{i\hbar}{2\rho_0} (\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^*) = 0. \quad (5.3)$$

The collective flow is assumed to be **irrotational**, i.e., $\nabla^2 \phi = 0$ which means $\nabla^2 \Psi = \nabla^2 \Psi^* = 0$ where the Laplacian in the spherical polar-coordinate is expressed as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}. \quad (5.4)$$

According to Jackiw [33, 34], the two Clebsch parameters ψ and λ are constructed as

$$\left. \begin{aligned} \lambda &= 2 \left(1 - \sin^2 \frac{f}{2} \sin^2 \theta \right), \quad \frac{\partial \lambda}{\partial \varphi} = 0, \\ \psi &= \varphi + \tan^{-1} \left(\tan \frac{f}{2} \cos \theta \right), \quad \frac{\partial \psi}{\partial \varphi} = 1, \end{aligned} \right\} \quad \frac{\partial \Psi}{\partial \varphi} = \sqrt{\frac{\rho_0\lambda}{\hbar}}, \quad (5.5)$$

where a profile function $f(=f(r))$ is determined later. Denoting the prime as differentiation with respect to r , the differential formulas for ψ and λ with respect to r and θ are given as

$$\left. \begin{aligned} \frac{\partial \lambda}{\partial r} &= \sin f \sin^2 \theta f', \\ \frac{\partial \Psi}{\partial r} &= -\sqrt{\frac{\rho_0\lambda}{\hbar}} \left\{ \frac{1}{2\lambda} \sin f \sin^2 \theta f' (\psi + i) - \frac{\cos \theta}{2(1 - \sin^2 \frac{f}{2} \sin^2 \theta)} f' \right\}, \\ \frac{\partial \psi}{\partial r} &= \frac{\cos \theta}{2(1 - \sin^2 \frac{f}{2} \sin^2 \theta)} f', \end{aligned} \right\} \quad (5.6)$$

$$\left. \begin{aligned} \frac{\partial \lambda}{\partial \theta} &= -2 \sin^2 \frac{f}{2} \sin 2\theta, \\ \frac{\partial \Psi}{\partial \theta} &= -\sqrt{\frac{\rho_0\lambda}{\hbar}} \left\{ \frac{1}{\lambda} \sin^2 \frac{f}{2} \sin 2\theta (\psi + i) + \frac{\sin f \sin \theta}{2(1 - \sin^2 \frac{f}{2} \sin^2 \theta)} \right\}, \\ \frac{\partial \psi}{\partial \theta} &= -\frac{\sin f \sin \theta}{2(1 - \sin^2 \frac{f}{2} \sin^2 \theta)}. \end{aligned} \right\} \quad (5.7)$$

Substitution of (5.5), (5.6) and (5.7) casts $\nabla^2\Psi=0$ into the following equation:

$$\begin{aligned}
\nabla^2\Psi &= -\sqrt{\frac{\rho_0\lambda}{\hbar}} \left[\left\{ \frac{1}{\{2(1-\sin^2\frac{f}{2}\sin^2\theta)\}^2} \left(\frac{1}{4}\sin^2f\sin^4\theta f'^2 + \frac{4}{r^2}\sin^4\frac{f}{2}\sin^2\theta\cos^2\theta \right) \right. \right. \\
&+ \frac{1}{2} \frac{1}{2(1-\sin^2\frac{f}{2}\sin^2\theta)} \left(\cos f\sin^2\theta f'^2 + \sin f\sin^2\theta f'' + 2\sin f\sin^2\theta \frac{1}{r}f' + \frac{4}{r^2}\sin^4\frac{f}{2}(3\cos^2\theta-1) \right) \Big\} (\psi+i) \\
&- \frac{\cos\theta}{\{2(1-\sin^2\frac{f}{2}\sin^2\theta)\}^2} \left\{ \sin f\sin^2\theta f'^2 + 2(1-\sin^2\frac{f}{2}\sin^2\theta)f'' + 4(1-\sin^2\frac{f}{2}\sin^2\theta)\frac{1}{r}f' - \frac{4}{r^2}\sin f \right\} \\
&+ \left. \frac{\cos\theta}{\{2(1-\sin^2\frac{f}{2}\sin^2\theta)\}^2} \left\{ \sin f\sin^2\theta f'^2 - \frac{4}{r^2}\sin f\sin^2\frac{f}{2}\sin^2\theta \right\} \right] \\
&= -\sqrt{\frac{\rho_0\lambda}{\hbar}} \left[\left\{ \frac{1}{\{2(1-\sin^2\frac{f}{2}\sin^2\theta)\}^2} \left((1-\sin^2\frac{f}{2}\sin^2\theta)^2 - \cos^2\theta \right) f'^2 + \frac{4}{r^2}\sin^4\frac{f}{2}(1-\sin^2\frac{f}{2})\sin^4\theta \right) \right. \\
&+ \left. \frac{\sin^2\theta}{4(1-\sin^2\frac{f}{2}\sin^2\theta)} \left(\sin f f'' + 2\sin f \frac{1}{r}f' + \frac{8}{r^2}\sin^4\frac{f}{2}\frac{\cos^2\theta}{\sin^2\theta} \right) \right\} (\psi+i) - \frac{\cos\theta}{2(1-\sin^2\frac{f}{2}\sin^2\theta)} \left\{ f'' + \frac{2}{r}f' - \frac{2}{r^2}\sin f \right\} \Big] \\
&= -\sqrt{\frac{\rho_0\lambda}{4\hbar}} \left[\left\{ \frac{2\sin^4\frac{f}{2}(1-\sin^2\frac{f}{2})\sin^4\theta}{r^2\cos^2\theta} + \frac{\sin f\sin^2\theta}{2\cos\theta} \left(f'' + \frac{2}{r}f' \right) \right\} (\psi+i) - \left(f'' + \frac{2}{r}f' - \frac{2}{r^2}\sin f \right) \right] = 0.
\end{aligned} \tag{5.8}$$

In the above, to eliminate the term f'^2 , we have assumed an auxiliary condition

$$1 - \sin^2\frac{f}{2}\sin^2\theta = \cos\theta, \Rightarrow \text{solution: } \cos\theta=1 \text{ or } \cot^2\frac{f}{2}. \tag{5.9}$$

This is the first time that such a solution can be found. The condition and solution make an essential role to satisfy the Laplace equation $\nabla^2\Psi=0$. For the present central purpose, from the outset, we demand that a profile function $f(=f(r))$ obeys a differential equation $f'' + \frac{2}{r}f' - \frac{2}{r^2}\sin f = 0$. This equation is first required to determine f by Jackiw-Pi [34]. This is integrated numerically by Bergner (Ref. [10] of [34]). Introducing a dimensionless variable x as $x = \frac{r}{R_0}$, the positive solution is plotted in Fig.1 of [34]. They present the solution of $f(x)$ that is regular at the origin, $x=0$, vanishing linearly with $x(0 < x < 1)$, and tending to π in an oscillatory manner for larger x . At the region $0 < x < 1$, we have $\sin f \approx 0$ and due to (5.9) $\cos\theta=1$. The coefficient of the term $(\psi+i)$ in the last line of equation (5.8) becomes vanishing accurately. Thus we obtain almost complete solution for $\nabla^2\Psi=0$ over the whole region of x .

By projecting onto a fixed direction \hat{n}^a (constant unit vector; $a=1\sim 3$) in the isospin space and using $U^{-1}dU$, consider a Clebsch-parameterized gauge potential 1-form a [33, 34] given as

$$a = i\text{Tr}\hat{n}^a\sigma^a U^{-1}dU, \text{ (use of summation convention in } \hat{n}^a\sigma^a), \tag{5.10}$$

where $U = e^{\frac{\hat{\omega}^a\sigma^a f}{2i}} = \cos\frac{f}{2} - i\sigma^a\hat{\omega}^a\sin\frac{f}{2}$, $\hat{\omega}^a \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and σ^a is the Pauli matrices.

Taking \hat{n}^a to point in the third direction, the 1-form a (5.10) is expressed as

$$a = \cos\theta df - \sin f \sin\theta d\theta - (1 - \cos f)\sin^2\theta d\varphi, \text{ (see Appendix B)}. \tag{5.11}$$

The $\frac{1}{24\pi^2} \int \text{Tr}(U^{-1}dU)^3$ is an integer, i.e., the winding number $\alpha_{i,j}$ or the quantized helicity [35, 34] related to the Chern-Simons numbers of a non-Abelian vacuum gauge potential [36]. For motions of quantum vortices, Nambu set up Hamilton-Jacobi formalism and reached a conclusion that the corresponding Hamilton-Jacobi functions are the Clebsch potentials [37]. While, the canonical quantization in terms of the Clebsch parameters has been developed by Rasetti-Regge. Their commutation relations are those of a current algebra [38, 39, 40].

6 Discussions and further outlook

In the preceding sections, we have attempted at a description of **rotational** velocity field of fluid in nuclei through the Clebsch transformation. Going to quantum fluid dynamics, we have derived a **vortex** Hamiltonian of the fluid in terms of roton operators. Under the previous preliminaries, the quantum fluid-dynamical approach may be applied to the three-dimensional nuclear fluid. Such an application to nuclei will provide an excellent description of another kind of elementary energy excitation, namely, "**vortex mode**" because such the approach is designed to take into account essential many-body effects, which were not considered in the traditional treatments of the problem of rotational collective motion.

While, extending Tomonaga's idea and using Sunakawa method, one of the present authors (S.N.) has developed collective description of nuclear surface oscillations and collective theory of two-dimensional nuclei, which have been made obeying the first quantized language, contrary to the second quantized manner adopted in the Sunakawa method. Applying the Tomonaga's revolutionary idea and the Sunakawa's discrete integral equation method for collective theory, we have developed successfully an *exact* canonically momenta approach to one-dimensional neutron-proton systems and a velocity operator approach to three-dimensional neutron-proton systems [41]. Particularly in the latter approach, we, however, have restricted Hilbert space to subspace in which eigenvalue of the **vortex operator** satisfies $\text{rot}\mathbf{v}(\mathbf{x})| \gg 0$, where the velocity operator $\mathbf{v}(\mathbf{x})$ is given as $\mathbf{v}(\mathbf{x}) = -\nabla\phi(\mathbf{x}) + \lambda(\mathbf{x})\nabla\psi(\mathbf{x})$, i.e., (2.1).

To describe a **rotational** velocity field of a fluid in nuclei, we have introduced the Clebsch transformation. Nevertheless, we have no essential meanings of **vortex motion** more, which are important but unsolved problems. There are some plans for clues to such the problems:

Firstly, the vorticity w (2.2) becomes $\frac{i\hbar}{2\rho}(\nabla\Psi \times \nabla\Psi^* - \nabla\Psi^* \times \nabla\Psi)$. Since w is orthogonal with vectors $\nabla\lambda$ and $\nabla\psi$, if we add a condition $(\nabla\lambda) \cdot (\nabla\psi) = 0$, three vectors $w, \nabla\lambda$ and $\nabla\psi$ are orthogonal with each other. Under the condition, the absolute value of w , $|w|$ becomes maximum. Due to the Ziman transformation and complex field operators Ψ and Ψ^* , that condition is rewritten as $\frac{i\hbar}{\rho}(\nabla\Psi - \nabla\Psi^*) \cdot \frac{\Psi\nabla\Psi^* - \Psi^*\nabla\Psi}{\Psi - \Psi^*}$. Then, $(\nabla\lambda) \cdot (\nabla\psi) = 0$ is changed to

$$\begin{aligned} & (\nabla\Psi - \nabla\Psi^*) \cdot (\Psi\nabla\Psi^* - \Psi^*\nabla\Psi) = 0 \\ & = (\nabla\Psi) \cdot \Psi\nabla\Psi^* - (\nabla\Psi^*) \cdot \Psi\nabla\Psi^* - (\nabla\Psi) \cdot \Psi^*\nabla\Psi + (\nabla\Psi^*) \cdot \Psi^*\nabla\Psi. \end{aligned} \quad (6.1)$$

The first term $(\nabla\Psi) \cdot \Psi\nabla\Psi^* (= \Xi)$ vanishes, due to $\langle 2020|10 \rangle = 0$, since Ξ has a form given below

$$\Xi \propto \langle 2020|10 \rangle R_0 \sum_{\mu\mu'\mu''k} \langle 2\mu 1k | 1\mu+k \rangle \langle 1\mu+k 2\mu' | 1\mu''+k \rangle \langle 2-\mu'' 1-k | 1-\mu''-k \rangle b_{2\mu} b_{2\mu'} b_{2\mu''}^* = 0. \quad (6.2)$$

The other terms also vanish. Then the above orthogonal condition is automatically satisfied. Eckart discussed an extension of the vorticity w to $\nu(\nabla\lambda) \times (\nabla\psi)$, where ν is some scalar function. The ν , however, can be taken equal to one without loss of generality [42].

Secondly, consider the continuity equation of fluid, $\dot{\rho} + \text{div}(\rho\mathbf{v}) = \dot{\rho} + (\mathbf{v} \cdot \nabla)\rho + \rho \text{div}\mathbf{v} = 0$, in which $\text{div}\mathbf{v} = 0$ does not necessarily require both $\dot{\rho} = 0$ and $\nabla\rho = 0$. If the Lagrange differentiation for density satisfies $\frac{D\rho}{Dt} \equiv \dot{\rho} + (\mathbf{v} \cdot \nabla)\rho = 0$, it leads to $\text{div}\mathbf{v} = 0$. While in the BMM, to get a surface-phonon state by quantization through $\mathbf{v} = -\nabla\phi$, the essential is a canonical commutation relation for ϕ and ρ , $[\phi(\mathbf{x}), \rho(\mathbf{x}')] = i\hbar\delta(\mathbf{x} - \mathbf{x}')$, which is given by the facts that the collective coordinate is expanded around a nuclear equilibrium and that the expansion coefficient of ϕ becomes $\dot{\alpha}$, i.e., π . Since we assume a constant density, $\rho = \rho_0$, we can not see an apparent role of the above commutation relation. To make further development of the BMM, we naturally reach an idea of the expansion of ρ around ρ_0 , i.e., $\rho = \rho_0 + \rho'$ [43, 17, 18, 20].

Lastly, we prove the gauge invariance of the velocity operator \mathbf{v} given in terms of the Clebsch parameters as $\mathbf{v} = -\nabla\phi - \frac{1}{\rho}\pi\nabla\psi$, i.e., (2.1) where we have used the relation $\pi = -\rho\lambda$.

We consider the following gauge transformation with the generating function $\omega \left(= \omega \left(\frac{\pi'}{\rho}, \psi \right) \right)$:

$$\phi' = \phi + \pi' \frac{\partial\omega}{\partial\pi'} - \omega, \quad \pi' = \pi + \rho \frac{\partial\omega}{\partial\psi}, \quad \psi' = \psi - \rho \frac{\partial\omega}{\partial\pi'}, \quad \rho \frac{\partial\omega}{\partial\rho} + \omega + \pi' \frac{\partial\omega}{\partial\pi'} - \omega = 0. \quad (6.3)$$

This kind of the gauge transformation was proposed by Ito [44, 19]. Kambe also has formulated a variation of ideal fluid flows according to the gauge principle in the Clebsch solution [45]. Substituting the gauge transformation (6.3) into $\mathbf{v}' = -\nabla\phi' - \frac{1}{\rho'}\pi'\nabla\psi'$, we calculate it as

$$\mathbf{v}' = -\nabla \left(\phi + \pi' \frac{\partial\omega}{\partial\pi'} - \omega \right) - \frac{1}{\rho} \left(\pi + \rho \frac{\partial\omega}{\partial\psi} \right) \nabla \left(\psi - \rho \frac{\partial\omega}{\partial\pi'} \right) = -\nabla\phi - \frac{1}{\rho}\pi\nabla\psi = \mathbf{v}, \quad (6.4)$$

where we have used the second and last relations of (6.3) and the gradient formula for ω [19],

$$\nabla\omega = \frac{\partial\omega}{\partial\psi}\nabla\psi + \frac{\partial\omega}{\partial\pi'}\nabla\pi' - \frac{1}{\rho}\pi' \frac{\partial\omega}{\partial\pi'}\nabla\rho. \quad (6.5)$$

Thus we prove the gauge invariance of the velocity \mathbf{v} . Following [19], separate \mathbf{v} as $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$.

Putting (6.3) into velocity $\mathbf{v}'_0 \left(= -\nabla\phi' - \frac{1}{\rho_0}\pi'\nabla\psi' \right)$ and using (6.5), we compute it as

$$\begin{aligned} \mathbf{v}'_0 &= -\nabla \left(\phi + \pi' \frac{\partial\omega}{\partial\pi'} - \omega \right) - \frac{1}{\rho_0} \left(\pi + \rho \frac{\partial\omega}{\partial\psi} \right) \nabla \left(\psi - \rho \frac{\partial\omega}{\partial\pi'} \right) \\ &= -\nabla\phi - \frac{1}{\rho_0}\pi\nabla\psi - \frac{\rho'}{\rho_0} \left(\frac{\partial\omega}{\partial\psi}\nabla\psi - \pi' \nabla \frac{\partial\omega}{\partial\pi'} - \frac{1}{\rho}\pi' \frac{\partial\omega}{\partial\pi'}\nabla\rho \right) \\ &= -\nabla\phi - \frac{1}{\rho_0}\pi\nabla\psi - \frac{\rho'}{\rho_0} \nabla \left(\omega - \pi' \nabla \frac{\partial\omega}{\partial\pi'} \right) = \mathbf{v}_0 - \frac{\rho'}{\rho_0} \nabla(\phi - \phi'), \end{aligned} \quad (6.6)$$

where linearization of $\frac{1}{\rho}$ is made as $\frac{1}{\rho} \approx \frac{1}{\rho_0} \left(1 - \frac{\rho'}{\rho_0} + \frac{\rho'^2}{\rho_0^2} \dots \right)$ and the first relation of (6.3) is used.

Another velocity \mathbf{v}_1 is expressed as $\mathbf{v}_1 = \frac{\rho'}{\rho_0} \left(\frac{1}{\rho_0} + \dots \right) \pi \nabla\psi$. Since the gauge invariance of the velocity \mathbf{v} is guaranteed, the velocities \mathbf{v}_0 and \mathbf{v}_1 must be invariant, respectively. From the R.H.S. in the last line of the equation (6.6), we must demand $\nabla(\phi - \phi') = 0$ which means

$$\nabla\omega - \nabla\pi' \frac{\partial\omega}{\partial\pi'} - \pi' \nabla \frac{\partial\omega}{\partial\pi'} = 0. \quad \text{Further, using this relation and (6.5), } \pi' \nabla\psi' \text{ is calculated as}$$

$$\pi' \nabla\psi' = \left(\pi + \rho \frac{\partial\omega}{\partial\psi} \right) \nabla \left(\psi - \rho \frac{\partial\omega}{\partial\pi'} \right) = \pi \nabla\psi + \frac{\rho}{\pi'} \left(\pi' - \pi - \rho \frac{\partial\omega}{\partial\psi} \right) \frac{\partial\omega}{\partial\psi} \nabla\psi = \pi \nabla\psi. \quad \text{Then, } \mathbf{v}'_1 = \mathbf{v}_1 \text{ is proved.}$$

Thus, we can also prove the gauge invariance of the velocity \mathbf{v} , even if we make the separation of \mathbf{v} as $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ under the linearization of $\frac{1}{\rho}$ around $\frac{1}{\rho_0}$ as given in the above.

Contrary to the present ways to the vortex motion in nuclei, we notice the papers [46, 47] in which Holtzwarth, Schütte and Eckart have attempted at derivation of fluid-dynamical equations of motion allowing for velocity fields with vorticity. They have extended a modulus and a phase fermion system in a many-body wave function of a fermion system to include two-body terms and given a time-dependent variational derivation of nuclear fluid dynamics of the system. Then they have derived an interesting relation between the vorticity and the two-body correlation, $\rho(\mathbf{x}) \frac{D\lambda(\mathbf{x}, t)}{Dt} = \int \rho^{(2)}(\mathbf{x}, \mathbf{x}') \frac{D\psi(\mathbf{x}', t)}{Dt} d\mathbf{x}'$ with $\int \rho^{(2)}(\mathbf{x}, \mathbf{x}') d\mathbf{x}' = (A-1)\rho^{(1)}(\mathbf{x})$. $\rho^{(2)}(\mathbf{x}, \mathbf{x}')$ and $\rho^{(1)}(\mathbf{x})$ are a two-body density and a usual one-body density, respectively. It is very important to study the above relation from the viewpoint of Ziman transformation for deeper understanding the vortex motion as a peculiar illustration of nuclear fluid dynamics.

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Appendix

A Unitary transformation of a kinetic operator

The unitary transformation of a kinetic operator $\frac{1}{2m}\mathbf{p}_n^2$ is calculated tediously but straightforwardly as

$$\begin{aligned}
U \frac{1}{2m} \mathbf{p}_n^2 U^{-1} &= \frac{1}{2m} \mathbf{p}_n^2 - \frac{1}{2m} \frac{\hbar}{i} \left\{ \mathbf{p}_n \frac{\partial U}{\partial \mathbf{x}_n} + \frac{\partial U}{\partial \mathbf{x}_n} \mathbf{p}_n \right\} U^{-1} \\
&= \frac{1}{2m} \mathbf{p}_n^2 + \frac{\mathbf{p}_n}{2m} \int \{ m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \phi(\mathbf{x}) - \lambda(\mathbf{x}) m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \psi(\mathbf{x}) \} d\mathbf{x} \\
&\quad + \frac{1}{2m} \int \{ m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \phi(\mathbf{x}) - \lambda(\mathbf{x}) m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \psi(\mathbf{x}) \} d\mathbf{x} \cdot U \mathbf{p}_n U^{-1} \\
&= \frac{1}{2m} \mathbf{p}_n^2 + \frac{\mathbf{p}_n}{2m} \int \{ m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \phi(\mathbf{x}) - \lambda(\mathbf{x}) m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \psi(\mathbf{x}) \} d\mathbf{x} \\
&\quad + \frac{1}{2m} \int \{ m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \phi(\mathbf{x}) - \lambda(\mathbf{x}) m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \psi(\mathbf{x}) \} d\mathbf{x} \\
&\quad \cdot \left[\mathbf{p}_n + \int \{ m \delta(\mathbf{x}' - \mathbf{x}_n) \nabla' \phi(\mathbf{x}') - \lambda(\mathbf{x}') m \delta(\mathbf{x}' - \mathbf{x}_n) \nabla' \psi(\mathbf{x}') \} d\mathbf{x}' \right] \\
&= \frac{1}{2m} \mathbf{p}_n^2 + m \int \frac{1}{2m} \{ \delta(\mathbf{x} - \mathbf{x}_n) + \delta(\mathbf{x} - \mathbf{x}_n) \mathbf{p}_n \} d\mathbf{x} \\
&\quad - m \int \lambda(\mathbf{x}) \frac{1}{2m} \{ \mathbf{p}_n \delta(\mathbf{x} - \mathbf{x}_n) + \delta(\mathbf{x} - \mathbf{x}_n) \mathbf{p}_n \} \cdot \nabla \psi(\mathbf{x}) d\mathbf{x} \\
&\quad + \frac{1}{2m} \int \{ m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \phi(\mathbf{x}) - \lambda(\mathbf{x}) m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \psi(\mathbf{x}) \} \\
&\quad \times \{ m \delta(\mathbf{x}' - \mathbf{x}_n) \nabla' \phi(\mathbf{x}') - \lambda(\mathbf{x}') m \delta(\mathbf{x}' - \mathbf{x}_n) \nabla' \psi(\mathbf{x}') \} d\mathbf{x} d\mathbf{x}'.
\end{aligned} \tag{A. 1}$$

We here omit the symmetrized term Sym. in (3.5). The last term of (A. 1) is calculated as

$$\begin{aligned}
&\int m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \phi(\mathbf{x}) m \delta(\mathbf{x}' - \mathbf{x}_n) \nabla' \phi(\mathbf{x}') d\mathbf{x} d\mathbf{x}' \\
&\quad - \int \lambda(\mathbf{x}) m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \psi(\mathbf{x}) m \delta(\mathbf{x}' - \mathbf{x}_n) \nabla' \phi(\mathbf{x}') d\mathbf{x} d\mathbf{x}' \\
&\quad - \int m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \phi(\mathbf{x}) \lambda(\mathbf{x}') m \delta(\mathbf{x}' - \mathbf{x}_n) \nabla' \psi(\mathbf{x}') d\mathbf{x} d\mathbf{x}' \\
&\quad + \int \lambda(\mathbf{x}) m \delta(\mathbf{x} - \mathbf{x}_n) \nabla \psi(\mathbf{x}) \lambda(\mathbf{x}') m \delta(\mathbf{x}' - \mathbf{x}_n) \nabla' \psi(\mathbf{x}') d\mathbf{x} d\mathbf{x}' \\
&= m \int \delta(\mathbf{x} - \mathbf{x}') \nabla \phi(\mathbf{x}) m \delta(\mathbf{x}' - \mathbf{x}_n) \nabla' \phi(\mathbf{x}') d\mathbf{x} d\mathbf{x}' \\
&\quad - m \int \delta(\mathbf{x} - \mathbf{x}') \nabla \psi(\mathbf{x}) m \delta(\mathbf{x}' - \mathbf{x}_n) \nabla' \phi(\mathbf{x}') d\mathbf{x} d\mathbf{x}' \\
&\quad - m \int \delta(\mathbf{x} - \mathbf{x}') \nabla \phi(\mathbf{x}) \lambda(\mathbf{x}') m \delta(\mathbf{x}' - \mathbf{x}_n) \nabla' \psi(\mathbf{x}') d\mathbf{x} d\mathbf{x}' \\
&\quad + m \int \delta(\mathbf{x} - \mathbf{x}') \lambda(\mathbf{x}) \nabla \psi(\mathbf{x}) \lambda(\mathbf{x}') m \delta(\mathbf{x}' - \mathbf{x}_n) \nabla' \psi(\mathbf{x}') d\mathbf{x} d\mathbf{x}',
\end{aligned} \tag{A. 2}$$

in which the first term becomes as,

$$m \int (\delta(\mathbf{x} - \mathbf{x}') \nabla \phi(\mathbf{x}) d\mathbf{x}) m \delta(\mathbf{x}' - \mathbf{x}_n) \nabla' \phi(\mathbf{x}') d\mathbf{x}' = m \int \nabla' \phi(\mathbf{x}') m \delta(\mathbf{x}' - \mathbf{x}_n) \nabla' \phi(\mathbf{x}') d\mathbf{x}'. \tag{A. 3}$$

The other terms are also similarly computed. Finally taking summation over n and using the formula for density operator $m \sum_n \delta(\mathbf{x}' - \mathbf{x}_n) = \rho(\mathbf{x}')$, we can reach the desired result.

B Detailed derivation of Eq. (5.11)

Using $U = \cos \frac{f}{2} - i\sigma^a \hat{\omega}^a \sin \frac{f}{2}$, $\hat{\omega}^a \equiv (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$ given in the previous Section 5, dU is calculated as

$$\left. \begin{aligned} dU &= -\frac{1}{2} \left(\sin \frac{f}{2} + i\sigma^a \hat{\omega}^a \cos \frac{f}{2} \right) df - i\sigma^a \sin \frac{f}{2} d\hat{\omega}^a, \\ d\hat{\omega}^a &= (\cos\theta d\theta \cos\varphi - \sin\theta \sin\varphi d\varphi, \cos\theta d\theta \sin\varphi + \sin\theta \cos\varphi d\varphi, -\sin\theta d\theta). \end{aligned} \right\} \quad (\text{B. 1})$$

Further using $U^{-1} = U^\dagger$ and (B. 1), $U^{-1}dU$ is computed as

$$\begin{aligned} U^{-1}dU &= \left(\cos \frac{f}{2} + i\sigma^a \hat{\omega}^a \sin \frac{f}{2} \right) \left\{ -\frac{1}{2} \left(\sin \frac{f}{2} + i\sigma^a \hat{\omega}^a \cos \frac{f}{2} \right) df - i\sigma^b \sin \frac{f}{2} d\hat{\omega}^b \right\} \\ &= -\frac{1}{4} (\sin f + 2i\sigma^b \hat{\omega}^b - \sigma^b \sigma^c \hat{\omega}^b \hat{\omega}^c \sin f) df - i\frac{1}{2} \sigma^b \sin f \hat{\omega}^b + \frac{1}{2} \sigma^b \hat{\omega}^b \sigma^c (1 - \cos f) d\hat{\omega}^c. \end{aligned} \quad (\text{B. 2})$$

We are now in the stage to compute $a = i\text{Tr} \hat{n}^a \sigma^a U^{-1}dU$ explicitly. The trace formulas $\text{Tr} \sigma^a = 0$ and $\text{Tr}(\sigma^a \sigma^b) = 2\delta_{ab}$ are useful. For our aim, let us prepare the following trace formulas:

$$\begin{aligned} -\frac{1}{4} i\text{Tr} \hat{n}^a \sigma^a (\sin f + 2i\sigma^b \hat{\omega}^b) df &= \frac{1}{2} \text{Tr}(\sigma^a \sigma^b) \hat{n}^a \hat{\omega}^b df = \delta_{ab} \hat{n}^a \hat{\omega}^b df = \hat{\omega}^3 df = \cos\theta df, \\ \frac{1}{2} \text{Tr} \hat{n}^a \sigma^a \sigma^b \sin f d\hat{\omega}^b &= \frac{1}{2} \text{Tr}(\sigma^a \sigma^b) \hat{n}^a \sin f d\hat{\omega}^b = \delta_{ab} \hat{n}^a \sin f d\hat{\omega}^b = \sin f \hat{\omega}^3 = -\sin f \sin\theta d\theta, \\ \text{Tr}(\sigma^a \sigma^b \sigma^c) \hat{n}^a \hat{\omega}^b \hat{\omega}^c &= \text{Tr}(\sigma^a \delta^{bc} + \sigma^a i\epsilon^{bcd} \sigma^b \sigma^c) \hat{n}^a \hat{\omega}^b \hat{\omega}^c = i\epsilon^{bcd} \text{Tr}(\sigma^a \sigma^d) \hat{n}^a \hat{\omega}^b \hat{\omega}^c \\ &= 2i\epsilon^{bca} \hat{n}^a \hat{\omega}^b \hat{\omega}^c = 2i\epsilon^{bc3} \hat{\omega}^b \hat{\omega}^c = 2i(\hat{\omega}^1 \hat{\omega}^2 - \hat{\omega}^2 \hat{\omega}^1) = 0, \\ \frac{1}{2} i\text{Tr} \hat{n}^a \sigma^a \sigma^b \hat{\omega}^b \sigma^c (1 - \cos f) d\hat{\omega}^c &= (1 - \cos f) \frac{1}{2} i \cdot 2i\epsilon^{bca} \hat{n}^a \hat{\omega}^b d\hat{\omega}^c = (1 - \cos f) \frac{1}{2} i \cdot 2i\epsilon^{bc3} \hat{\omega}^b d\hat{\omega}^c \\ &= (1 - \cos f) \frac{1}{2} i \{ 2(-i)(-1)(\hat{\omega}^1 d\hat{\omega}^2 - \hat{\omega}^2 d\hat{\omega}^1) \} \\ &= (1 - \cos f) \{ -\sin\theta \cos\varphi (\cos\theta d\theta \sin\varphi) - \sin\theta \cos\varphi (\sin\theta \cos\varphi d\varphi) \\ &\quad + \sin\theta \sin\varphi (\cos\theta d\theta \cos\varphi) + \sin\theta \sin\varphi (-\sin\theta \sin\varphi d\varphi) \} \\ &= (1 - \cos f) \sin^2 \varphi d\varphi, \end{aligned} \quad (\text{B. 3})$$

in which taking \hat{n}^a to point in the third direction. Gathering the above all formulas of (B. 3), at last we acquire the desired expression for a (5.11) as

$$a = \cos\theta df - \sin f \sin\theta d\theta - (1 - \cos f) \sin^2 \theta d\varphi. \quad (\text{B. 4})$$

As described by Jackiw-Pi [34], another formula for (B. 4) in the Clebsch representation for the velocity field \mathbf{v} , $\mathbf{v} = -\nabla\phi + \lambda\nabla\psi$, is given by

$$a = d(-2\varphi) + 2 \left(1 - \sin^2 \frac{f}{2} \sin^2 \theta \right) d \left\{ \varphi + \tan^{-1} \left(\tan \frac{f}{2} \cos \theta \right) \right\}. \quad (\text{B. 5})$$

Inversely, from (B. 5), we can easily derive (B. 4) by noticing the differentiation given below

$$d \left\{ \varphi + \tan^{-1} \left(\tan \frac{f}{2} \cos \theta \right) \right\} = d\varphi + \frac{1}{1 + \tan^2 \frac{f}{2} \cos^2 \theta} \left\{ \sec^2 \frac{f}{2} \left(\frac{f}{2} \right)' \cos \theta d\varphi - \tan \frac{f}{2} \sin \theta d\theta \right\}. \quad (\text{B. 6})$$

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